Math 60 9.3 & 9.4

Simplifying Using the Laws of Exponents

Objectives

9.3

1) Simplify expressions containing rational exponents using the laws of exponents.

2) Simplify radical expressions by converting to rational exponents and back to radicals

* 3) Factor expressions containing rational expressions

9.4

4) Multiply radicals with unlike indices by converting to rational exponents and back to radicals.

* STEM majors! You will need this when you get to calculus.

Math 60 9.3 + 9.4 – 2nd

Simplify.

This instruction has one word, but it means many instructions, which you must know are included, even when not stated.

#1) No parentheses in answer

#2) No negative exponents in answer — move them to make them positive

#3) Each base appears only once — use laws of exponents to combine

#4) No zero exponents — simplify.

#5) No powers raised to powers

#6) If question is given using exponents, final answer should use exponents.

#7) If question is given using radicals, final answer should be a simplified radical.

Math 60 Exponent Laws

Exponent law #1 $a^n \cdot a^m = a^{n+m}$ Example: $x^2 \cdot x^3 = x^{2+3} = x^5$

With fraction exponents, it works the same

- Same base, appears twice
- Multiplied bases
- Add exponents

$$a^{p/q} \cdot a^{r/t} = a^{p/q + r/t}$$

Fraction example: $x^{2/3} \cdot x^{1/6} = x^{2/3 + 1/6} = x^{5/6}$

Caution: Be patient! You know how to add fractions, but it may take an extra step of work, or careful use of your calculator.

Exponent law #2 $\dfrac{a^n}{a^m} = a^{n-m}$ Examples: $\dfrac{x^3}{x^2} = x^{3-2} = x^1 = x$

$$\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3} = x^{-3}$$

With fraction exponents, it works the same

- Same base, appears twice
- Divided bases
- Subtract exponents, numerator exponent minus denominator →result in numerator
- OR Subtract exponents, denominator exponent minus numerator →result in denominator

$$\frac{a^{p/q}}{a^{r/t}} = a^{p/q - r/t}$$

Fraction examples: $\dfrac{x^{2/3}}{x^{1/6}} = x^{2/3 - 1/6} = x^{3/6} = x^{1/2}$

$$\frac{x^{1/4}}{x^{3/4}} = \frac{1}{x^{3/4 - 1/4}} = \frac{1}{x^{2/4}} = \frac{1}{x^{1/2}}$$

Exponent law #3 $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ Examples: $x^{-3} = \dfrac{1}{x^3}$

$$\frac{1}{x^{-3}} = x^3$$

With fraction exponents, it works the same

- Negative exponent
- Move exponent from numerator to denominator (or denominator to numerator) of fraction, change sign of exponent to positive

$$a^{-p/q} = \frac{1}{a^{p/q}}$$

Fraction example: $x^{-2/3} = \dfrac{1}{x^{2/3}}$

$$\frac{1}{a^{-p/q}} = a^{p/q}$$

Fraction example: $\dfrac{1}{x^{-2/3}} = x^{2/3}$

Exponent law #4 $a^0 = 1$ Examples: $1 = \frac{x^2}{x^2} = x^{2-2} = x^0 = 1$

Fraction example: $1 = \frac{x^{2/3}}{x^{2/3}} = x^{2/3 - 2/3} = x^0 = 1$

Caution: Zero exponent resolves to the number 1. There is no base anymore!

Exponent law #5 $(a^n)^m = a^{n \cdot m}$ Example: $(x^{-3})^2 = x^{-3 \cdot 2} = x^{-6} = \frac{1}{x^6}$

With fraction exponents, it works the same

- One base
- Parentheses separate two exponents
- Multiply exponents

$\left(a^{p/q}\right)^{r/t} = a^{p/q \cdot r/t}$ Fraction example: $\left(x^{2/3}\right)^{1/6} = x^{2/3 \cdot 1/6} = x^{1/9}$

Exponent law #6 $(ab)^n = a^n b^n$ Examples: $(xy)^{-2} = x^{-2}y^{-2} = \frac{1}{x^2y^2}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ $\left(\frac{x}{y}\right)^{-3} = \left(\frac{y}{x}\right)^3 = \frac{y^3}{x^3}$

With fraction exponents, it works the same

- Two bases, inside parentheses
- One exponent outside parentheses
- "Share" exponents, each base inside parentheses gets the exponent outside those parentheses

$(ab)^{p/q} = a^{p/q} b^{p/q}$ Fraction examples: $(xy)^{2/3} = x^{2/3} y^{2/3}$

$\left(\frac{a}{b}\right)^{p/q} = \frac{a^{p/q}}{b^{p/q}}$ $\left(\frac{x}{y}\right)^{2/3} = \frac{a^{2/3}}{b^{2/3}}$

$\left(\frac{a}{b}\right)^{-p/q} = \left(\frac{b}{a}\right)^{p/q} = \frac{b^{p/q}}{a^{p/q}}$ $\left(\frac{x}{y}\right)^{-2/3} = \left(\frac{y}{x}\right)^{2/3} = \frac{y^{2/3}}{x^{2/3}}$

Caution: No math term exists for what we do with these exponents. Do not use the word "distribute", which refers specifically to multiplying a sum, such as $3(2+5) = 3 \cdot 2 + 3 \cdot 5$

Math 60 9.3

Simplify. Assume all variables are non-negative.

① $\left(4^{1/2}\, a^{-3/2}\, b^{1/4}\right)^8$

$= \left(\dfrac{4^{1/2}\, b^{1/4}}{a^{3/2}}\right)^8$ move neg exponent

$= \dfrac{\left(4^{1/2}\right)^8 \left(b^{1/4}\right)^8}{\left(a^{3/2}\right)^8}$ exponent law $(ab)^n = a^n b^n$, $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

$= \dfrac{4^{1/2\cdot 8}\; b^{1/4\cdot 8}}{a^{3/2\cdot 8}}$ exponent law $(a^n)^m = a^{n\cdot m}$

$= \dfrac{4^4\, b^2}{a^{12}}$

$$\begin{cases} \dfrac{1}{2}\cdot\dfrac{8}{1} = \dfrac{1}{1}\cdot\dfrac{4}{1} = 4 \\ \dfrac{1}{4}\cdot\dfrac{8}{1} = 2 \\ \dfrac{3}{2}\cdot\dfrac{8}{1} = \dfrac{3}{1}\cdot\dfrac{4}{1} = 12 \end{cases}$$

$= \boxed{\dfrac{256\, b^2}{a^{12}}}$ $4^4 = 256$

② $\left(32x^{-2}y^{-3}\right)^{2/3}\left(4x^2y^{1/2}\right)^{4/3}$

$= \left(\dfrac{32}{x^2y^3}\right)^{2/3}\left(4x^2y^{1/2}\right)^{4/3}$ move neg exponents $a^{-n} = \dfrac{1}{a^n}$

$= \dfrac{32^{2/3}}{(x^2)^{2/3}(y^3)^{2/3}} \cdot 4^{4/3}\cdot(x^2)^{4/3}(y^{1/2})^{4/3}$ exponent law $(ab)^n = a^n b^n$, $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

$= \dfrac{(2^5)^{2/3}(2^2)^{4/3}(x^2)^{4/3}(y^{1/2})^{4/3}}{(x^2)^{2/3}(y^3)^{2/3}}$ factor $32 = 2^5$, $16 = 2^4$

$= \dfrac{2^{5\cdot 2/3}\; 2^{2\cdot 4/3}\; x^{2\cdot 4/3}\; y^{1/2\cdot 4/3}}{x^{2\cdot 2/3}\; y^{3\cdot 2/3}}$ multiply exp $(a^n)^m = a^{n\cdot m}$

continued →

Math 60 9.3

continued

$$= \frac{2^{10/3} \cdot 2^{8/3} x^{8/3} y^{2/3}}{x^{4/3} y^2}$$

$$\begin{cases} \frac{5}{1} \cdot \frac{2}{3} = \frac{10}{3} \\ \frac{2}{1} \cdot \frac{4}{3} = \frac{8}{3} \\ \frac{2}{1} \cdot \frac{4}{3} = \frac{8}{3} \\ \frac{1}{2} \cdot \frac{4}{3} = \frac{1}{1} \cdot \frac{2}{3} = \frac{2}{3} \\ \frac{2}{1} \cdot \frac{2}{3} = \frac{4}{3} \\ \frac{3}{1} \cdot \frac{2}{3} = \frac{1}{1} \cdot \frac{2}{1} = 2 \end{cases}$$

$$= \frac{2^{10/3 + 8/3} \cdot x^{8/3 - 4/3}}{y^{2 - 2/3}}$$

exponent laws $a^n \cdot a^m = a^{n+m}$

$$\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$$

$$= \frac{2^6 \cdot x^{4/3}}{y^{4/3}}$$

$$\begin{cases} \frac{10}{3} + \frac{8}{3} = \frac{18}{3} = 6 \\ \frac{8}{3} - \frac{4}{3} = \frac{4}{3} \\ \frac{2}{1} - \frac{2}{3} = \frac{2 \cdot 3}{1 \cdot 3} - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \frac{4}{3} \end{cases}$$

$$= \boxed{\frac{64 x^{4/3}}{y^{4/3}}}$$

$2^6 = 64$

(3) $$\frac{\left(2^{1/2} x^{-1} y^{2/5}\right)^5}{2^{1/2} x^2 y^2}$$

Notice ( ) around numerator only means we cannot move x^{-1} yet. because it will move out of its parentheses.

$$= \frac{(2^{1/2})^5 (x^{-1})^5 (y^{2/5})^5}{2^{1/2} x^2 y^2}$$

exponent law $(ab)^n = a^n b^n$

$$= \frac{2^{1/2 \cdot 5} x^{-1 \cdot 5} y^{2/5 \cdot 5}}{2^{1/2} x^2 y^2}$$

exponent law $(a^n)^m = a^{n \cdot m}$

$$= \frac{2^{5/2} x^{-5} y^2}{2^{1/2} x^2 y^2}$$

$\frac{1}{2} \cdot \frac{5}{1} = \frac{5}{2}$

$\frac{2}{5} \cdot \frac{5}{1} = 2$

$-1 \cdot 5 = -5$

continued

Math 60 9.3

continued

$$= \frac{2^{5/2-1/2}\ y^{2-2}}{x^{2-(-5)}}$$

exponent law $\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$

$$\begin{cases} \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2 \\ 2-(-5) = 2+5 = 7 \end{cases}$$

$$= \frac{2^2\ y^0}{x^7}$$

$$= \frac{4 \cdot 1}{x^7}$$

$$= \boxed{\frac{4}{x^7}}$$

Math 60 9.3

Use rational exponents to simplify the radical expression. Assume all variables are non-negative.

(4) $\sqrt[6]{q^3}$

$= q^{3/6}$ write as rational (fraction) exponent

$= q^{1/2}$ reduce exponent

$= \sqrt{q}$ write as radical

$= \boxed{3}$

(5) $\sqrt[3]{27a^3b^9}$

$= (27a^3b^9)^{1/3}$ Write as rational exponent

$= 27^{1/3} \cdot (a^3)^{1/3} \cdot (b^9)^{1/3}$ Exponent Law $(ab)^n = a^n b^n$ share exp.

$= \sqrt[3]{27} \cdot a^{3 \cdot 1/3} \cdot b^{9 \cdot \frac{1}{3}}$ ~~Exp~~ Exp Law $a^n a^m = a^{n \cdot m}$ mult exp; write radical

$\frac{\cancel{3}}{1} \cdot \frac{1}{\cancel{3}} = 1$

$\frac{\cancel{9}^3}{1} \cdot \frac{1}{\cancel{3}_1} = 3$

$= 3 \cdot a^1 \cdot b^3$

$= \boxed{3ab^3}$

* check for even indices
Do we need absolute values?

(6) $\frac{\sqrt[4]{x^3}}{\sqrt{x}}$

$= \frac{x^{3/4}}{x^{1/2}}$ write with rational exp

$= x^{3/4 - 1/2}$ Exp Law $\frac{a^n}{a^m} = a^{n-m}$ subtract exp

$\frac{3}{4} - \frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{4}$

$= x^{1/4}$

$= \boxed{\sqrt[4]{x}}$ Question is a radical → final answer is a radical.

Math 60 9.3

7 $100^{0.25}$ Simplify without a calculator.

$= 100^{1/4}$ write exponent as fraction.

$= (10^2)^{1/4}$ write base with exponent

$= 10^{2 \cdot 1/4}$ exponent law $(a^n)^m = a^{n \cdot m}$

$= 10^{1/2}$ $\frac{2}{1} \cdot \frac{1}{4} = \frac{1}{2}$

$= \sqrt{10}$

$= \boxed{10^{0.5}}$ question had decimal exponent, write answer with decimal exponent

Math 60 9.3

8) $\sqrt{\sqrt[3]{n}}$

$= \left(n^{1/3}\right)^{1/2}$ write with rational exp

$= n^{1/3 \cdot 1/2}$ Exp Law #4 mult exp

$= n^{1/6}$

$= \boxed{\sqrt[6]{n}}$ question is a radical, so write final answer as a radical

Review

9) Factor out x^2:

$\underbrace{9x^3}_{\text{1st term}} + \underbrace{4x^2(2x+1)}_{\text{2nd term}}$ Both terms have x^2

$= x^2\left[9x + 4(2x+1)\right]$ Factor out x^2. Add brackets.

Distribute inside brackets

$= x^2\left[9x + 8x + 4\right]$ combine like terms inside brackets.

$= \boxed{x^2(17x+4)}$

STEM Majors!!

* This skill is used in calculus!

10) Factor out $x^{1/2}$

$3x^{3/2} + 2x^{1/2}(4x+1)$

when we factor out $x^{1/2}$, we are dividing $x^{1/2}$ out

$\frac{x^{3/2}}{x^{1/2}} = x^{3/2 - 1/2} = x^{2/2} = x^1 = x$

$= x^{1/2}\left[3x + 2(4x+1)\right]$

$= x^{1/2}\left[3x + 8x + 1\right] = \boxed{x^{1/2}(11x+1)}$

Math 60 9.3

(11) Factor out $x^{-1/3}$. Assume all variables are non-negative.

$$2x^{2/3} + x^{-1/3}(3x+2)$$

when we factor out $x^{-1/3}$, we are dividing $x^{-1/3}$ out

$$\frac{x^{2/3}}{x^{-1/3}} = x^{2/3-(-1/3)} = x^{2/3+1/3} = x^{3/3} = x^1 = x.$$

$$= x^{-1/3}\left[2x + (3x+2)\right]$$

$$= x^{-1/3}\left[2x + 3x + 2\right]$$

$$= x^{-1/3}(5x+2)$$

$$= \boxed{\frac{5x+2}{x^{1/3}}}$$

Distribute to simplify

(12) $x^{1/3}(x^{5/3}+4)$

$$= \underbrace{x^{1/3} \cdot x^{5/3}} + 4 \cdot x^{1/3}$$

$$= x^{1/3+5/3} + 4x^{1/3}$$

$$= x^{6/3} + 4x^{1/3}$$

$$= \boxed{x^2 + 4x^{1/3}}$$

(13) $3a^{-1/2}(2-a)$ Assume all variables are non-negative.

$$= 3a^{-1/2} \cdot 2 - 3a^{-1/2} \cdot a$$

$$a^{-1/2+1} = a^{-1/2+2/2} = a^{1/2}$$

$$= 3 \cdot 2 \cdot a^{-1/2} - 3a^{-1/2} \cdot a^1$$

$$= 6a^{-1/2} - 3a^{1/2}$$

$$= \boxed{\frac{6}{a^{1/2}} - 3a^{1/2}}$$

Math 60 9.4 unlike indices

Multiply and simplify

(14) $\sqrt[3]{5} \cdot \sqrt[4]{2}$

Notice: • indices are different
• no property of radicals currently applies
• must use exponents.

$= 5^{1/3} \cdot 2^{1/4}$

"Multiply" means we want one radical $\Rightarrow$ need an index we can use for both

$\Rightarrow$ write each fraction with common denominator

$\frac{1}{3} = \frac{4}{12}$

$\frac{1}{4} = \frac{3}{12}$

$= 5^{4/12} \cdot 2^{3/12}$

$= \sqrt[12]{5^4} \cdot \sqrt[12]{2^3}$

write as radicals with exponent on inside

$= \sqrt[12]{5^4 \cdot 2^3}$

use product property!

check if it can be simplified
- get prime factors
- need a power greater than or equal to index

(can't simplify)

$= \sqrt[12]{625 \cdot 8}$

multiply radicands

$= \boxed{\sqrt[12]{5000}}$

(15) $\sqrt[4]{8} \cdot \sqrt{5}$

$= 8^{1/4} \cdot 5^{1/2}$

CD = 4 $\quad \frac{1}{4}, \frac{2}{4}$

$= 8^{1/4} \cdot 5^{2/4}$

$= \sqrt[4]{8^1} \cdot \sqrt[4]{5^2}$

$= \sqrt[4]{2^3} \cdot \sqrt[4]{5^2}$

prime factors (exponents all less than index so it won't simplify)

$= \sqrt[4]{2^3 \cdot 5^2}$

$= \boxed{\sqrt[4]{200}}$

Math 60 9.3 – 2nd & 9.4 unlike indices.

(16) $\sqrt{2} \cdot \sqrt[3]{20}$

write with exponents

$= 2^{1/2} \cdot 20^{1/3}$

find common denominator $\quad \frac{1}{2} = \frac{3}{6} \quad \frac{1}{3} = \frac{2}{6}$

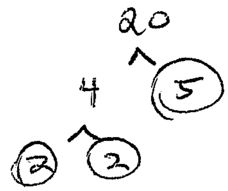
$= 2^{3/6} \cdot 20^{2/6}$

write as radicals, put exponents inside

$= \sqrt[6]{2^3} \cdot \sqrt[6]{20^2}$

product property

$= \sqrt[6]{2^3 \cdot 20^2}$



prime factors

$= \sqrt[6]{2^3 \cdot 2^4 \cdot 5^2}$

$20 = 2^2 \cdot 5$

$20^2 = (2^2)^2 (5)^2 = 2^4 \cdot 5^2$

add exponents

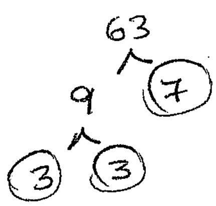
$= \sqrt[6]{2^7 \cdot 5^2}$

simplify

$= \sqrt[6]{2^6} \cdot \sqrt[6]{2 \cdot 5^2}$

$= \boxed{2\sqrt[6]{50}}$

(17) $\sqrt{3} \cdot \sqrt[3]{63}$



$63 = 3^2 \cdot 7$

$(63)^2 = 3^4 \cdot 7^2$

$= 3^{1/2} \cdot 63^{1/3}$

$= 3^{3/6} \cdot 63^{2/6}$

$= \sqrt[6]{3^3} \cdot \sqrt[6]{63^2}$

$= \sqrt[6]{3^3 \cdot 63^2}$

$= \sqrt[6]{3^3 \cdot 3^4 \cdot 7^2}$

$= \sqrt[6]{3^7 \cdot 7^2} = \sqrt[6]{3^6} \cdot \sqrt[6]{3 \cdot 7^2} = \boxed{3\sqrt[6]{147}}$

Math 60 9.3

Extra Practice / Review

Simplify

(18) $\dfrac{4^{2/3}}{4^{-5/6}}$

$= 4^{2/3-(-5/6)}$ exponent law $\dfrac{a^n}{a^m} = a^{n-m}$

$= 4^{2/3+5/6}$ $\dfrac{2}{2}\cdot\dfrac{2}{3}+\dfrac{5}{6} = \dfrac{9}{6} = \dfrac{3}{2}$

$= 4^{3/2}$

$= (\sqrt{4})^3$

$= 2^3$

$= \boxed{8}$

(19) $(4^{3/2})^{5/3}$

$= 4^{3/2\cdot 5/3}$ exponent law $(a^m)^n = a^{m\cdot n}$

$= 4^{5/2}$ $\dfrac{\cancel{3}}{2}\cdot\dfrac{5}{\cancel{3}} = \dfrac{5}{2}$

$= (\sqrt{4})^5$

$= 2^5$

$= \boxed{32}$